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A PROBABILITY ANALYSIS OF THE PROBLEM OF THE BREAKTHROUGH OF A COARSELY CELLULAR FOAM IN A POROUS MEDIUM†

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The problem of the carrying capacity of a system of film membranes (lamellae) in a porous medium, which block the free motion of a gas delivered from outside, is considered in a probability formulation. It is assumed that, in the initial state, the lamellae are only located in the pore throats. The system can be called a coarsely cellular foam since exactly one of its bubbles is actually enclosed in each pore. The lamellae are assumed to be immobile and are broken down if the pressure drop across them exceeds a certain critical value. The characteristics of the breakthrough cluster, which arises when a specified quantity of an ideal gas is injected into the medium, are investigated. The problem is formulated in terms of probability mechanics for the breakdown of discrete systems and is studied within the framework of lattice models. The pores (which, for simplicity, are of the same volume) are identified with lattice nodes and the lamellae are identified with links which are blocked in the initial state and which possess a random strength with a known probability distribution. The breakthrough process involves the successive rupture of overloaded lamellae and a corresponding enlargement of the domain of the pore space occupied by the injected gas. Analytic expressions for the probability of breakthrough to a specified depth are obtained for several forms of linear chains of lattice nodes and in the case when the structure of the system is a regular binary tree (a Cayley tree). Examples of calculations are presented. Since the probability of the lamellae breaking down decreases as the breakthrough zone increases, the model considered is substantially different from traditional percolation models and, in particular, the breakthrough cluster is always bounded here.

Interest in the problem of the behaviour of foams in a porous medium is due to their ability to block the motion of gas streams in the system of capillaries effectively. The corresponding blocking mechanisms are extremely diverse and include the effects of the size reduction of the structure of the foam, the breakdown and regeneration of lamellae, the displacement of the foam in the form of a caravan of lamellae along certain active pore channels, etc. (for example, see [1–3] and the bibliography in them). The key parameter of the problem is the critical pressure gradient at which steady gas flow occurs [1, 2] or the critical pressure drop corresponding to the instant when dynamic displacement of the lamellae of the caravan occurs, when an active channel informed [3–5]. In the second case, one speaks of the problem of the breakthrough of the foam.

In the first of the above-mentioned formulations, the problem is usually treated within the framework of percolation approaches (probabilistic by its nature) [1, 2, 6] when the geometrical characteristics of the clusters formed by the active channels are used under the assumption that there are no collective effects. In this case, the probability of a given link being open is assumed to be one and the same (the probabilities of their breakdown are assumed to be identical and constant for the whole ensemble of lamellae). The use of numerical simulation enables one to complicate models of this type by, for example, taking into account the effects of the repeated breakdown and regeneration of the lamellae [2]. The problem of breakthrough has been considered in [3–5] in a non-percolation, deterministic formulation, where a model was proposed for the displacement of the caravan of intact lamellae along a linear channel, which takes account of the property of the compressibility of the gas and the bubbles of foam and, thereby, the interaction of neighbouring lamellae. It has been shown that collective effects can have a substantial effect on the value of the critical pressure drop.

Below, the problem of the breakthrough of a foam is considered, but from a somewhat different aspect. In fact, a probability model is investigated which describes the formation of a breakthrough zone (or a breakthrough cluster according to the terminology adopted in percolation theory) when a specified quantity of an ideal gas is injected into the medium. The problem involves determining the parameters of the breakthrough zone, that is, it is not assumed that there is a gas flow. The successive breakdown

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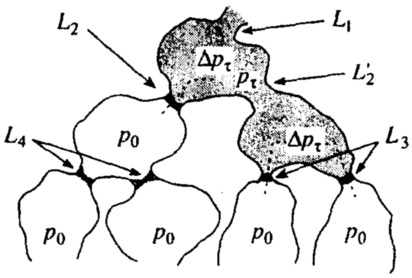

Fig. 1

of the blocking lamellae is accepted as the mechanism causing the growth of the breakthrough cluster: the possibility of their displacement or movement along the system of capillaries is not considered in the model. The reduction in the pressure in the breakthrough zone, as it becomes larger leads to a reduction in the likelihood of the breakdown of lamellae at the later stages of the breakthrough process compared with the initial stage, which distinguishes the model under consideration from traditional percolation formulations.

1. FORMULATION OF THE PROBLEM

We will formulate the problem of estimating the "carrying capacity" of an ensemble of lamellae which overlap capillaries within the framework of the following lattice model which, as will be shown below, can be investigated using the methods of probability mechanics for the breakdown of discrete systems. The lattice nodes are identified with pores, the volumes of which (v_0) are identical, and the links, which are blocked in the initial state, are identified with lamellae located in the pore throats. It is assumed that the medium is filled with a coarsely cellular foam (Fig. 1) when there is just a single lamella in each pore throat (L_i in Fig. 1) and each pore actually contains just a single bubble of the foam, and the pressures in these pores are identical and equal to p_0.

The lamellae are assumed to be immobile and are ruptured, opening the link, if the pressure drop on them exceeds a certain critical value, which can be called the strength s of the lamella. The strengths of lamellae are assumed to be independent random quantities with the same probability distribution $F_s(x)$, specified on the support $[s_{\min}, s_{\max}]$. An opened link cannot be blocked again, that is, regeneration of lamellae is not permitted.

A gas (which is the same as that in the pores) is injected at a specified lattice node which is henceforth called the reference lattice node. Suppose the total mass M of the injected gas is specified. The lamellae closest to the reference lattice node are acted upon by an excess pressure and some of them (the relatively weaker ones) are ruptured (the lamellae L_1 in Fig. 1) while other remain intact. The volume of the breakthrough zone increases and pressure equalization at a new lower level occurs. Again, some of the loaded lamellae are ruptured (the lamellae L'_2), a corresponding redistribution of the pressure takes place and so on. The breakthrough process is completed when all the lamellae in the breakthrough zone front are able to absorb the corresponding pressure drop (the lamellae L_2, L_3).

All time effects are excluded from the treatment and the breakthrough process is analysed using a step-by-step scheme. In the course of a single step, the synchronous breakdown of all the overloaded lamellae occurs and there is an equalization of the pressure within the limits of the new configuration of the breakthrough zone. The case of non-interacting lamellae is considered when the pressure perturbation is not transmitted beyond the limits of the breakthrough zone.

Using the model of an ideal gas, we find the following relation for the magnitude of the pressure in the breakthrough zone at step τ.

$$p_\tau = (W/V_\tau + 1)p_0$$

where W is the volume which the injected gas would occupy under the same conditions as the gas in the pores exists (the values of W and M are uniquely related) and V_τ is the volume of the breakthrough zone at the end of step τ. The pressure jump on the loaded lamellae is then equal to

$$\Delta p_\tau = (W/V_\tau)p_0 \tag{1.1}$$

Here, the probability of breakdown of a lamella (that is, the probability of the unblocking of a link) is equal to $F_s(\Delta p_\tau)$.

The following important property of the model follows from what has been said: if a lamella is not ruptured in a certain step, then it will never be ruptured later (the lamella L_2 in Fig. 1) since the pressure in the breakthrough zone can only decrease.

Within this formulation of the problem, the final shape of the breakthrough zone is random. The model for the breakthrough of a foam by a gas, which has been described, is largely analogous to the models which are investigated in percolation theory [6–8]. However, it differs considerably from these latter models in the fact that the probability of the unblocking of links in it decreases as the breakthrough zone (or the cluster, as one says in percolation theory) becomes larger while, in percolation models, this probability is fixed. One might therefore expect a considerable difference in the properties of the above-mentioned models. For example, it is obvious that, for a fixed amount of injected gas, the breakthrough cluster always has finite dimensions, while the effect of the occurrence of an infinite cluster is a key factor in percolation models.

A breakthrough cluster can be treated as a configuration of a set of ruptured elements of a certain structural system (here, a system of lamellae), which is acted upon by a specified load (the injection of a specified amount of gas) subject to the condition that the law for the redistribution of the loads on the working elements of the system is known (relation (1.1)). Hence, the problem of determining the probability of the breakthrough of a form at a specified depth can be treated using the methods of probability mechanics for the breakdown of discrete systems or the structural theory of reliability (for example, see [9, 10]†).

We shall understand a system of lamellae of order N to be a set of lamellae which are a distance of no more than N "steps" from the reference lattice node. We call the set of lattice nodes which are a distance of exactly k steps from the reference lattice node the kth structural level of this system (in regular lattices, such a set of lattice nodes is called the kth coordination group [7]). It follows that the "carrying capacity" (or strength) of a system of lamellae is understood to be their resistance to the process of enlargement of the breakthrough zone. We shall say that a system of lamellae of order N is broken down under a "load" W if at least one of the lattice nodes of the Nth structural level falls within the breakthrough zone. In this case, we shall also say that the breakthrough depth is no less than N.

A numerical characteristic of the strength of a system of lamellae R_N can be introduced which, as follows from what has been said above, is equal to the minimum value of the quantity W for which the breakthrough reaches a depth N. By virtue of the probability formulation of the problem, R_N is a random quantity and possesses a certain distribution function $F_R(x)$. According to the definition of the distribution function

$$F_R(W) = \Pr\{R_N < W\}$$

where the probability of an event, which consists of the fact that the strength of the system of lamellae is less than the applied load W, is written on the right-hand side (we recall that the quantity W is proportional to the amount of gas injected).

Thus, in order to characterize the strength of a system of lamellae, it is necessary to find the distribution function $F_R(x)$ or, what is the same thing, to find the set of values $P_N(W)$, that is, the breakthrough probabilities at a depth N for arbitrary values of W.

Remark. The system which is analysed within the framework of the model described belongs to the so-called class of systems with unloading since the load on the unruptured lamellae is reduced as the failure process in the system develops. In such systems, the breakthrough probability can depend on a succession of breakdowns of the elements [10]†. The version adopted here of the synchronous rupture of all of the lamellae which are overloaded in a given step is not the only possible one. However, an analytic investigation of other versions (successive rupture of lamellae in order of increasing values of their strength, for example) encounters significant difficulties.

If a certain actual topology of the lattice (pore system) is now specified, the assumptions which have been made are sufficient to enable the breakthrough problem to be analysed using one or other numerical simulation scheme (see [2], for example). The use of Monte-Carlo methods [11] makes a probability representation of the resulting parameters of the problem (the breakthrough depth or the volume of the breakthrough cluster) possible. However, it is only realistic to expect to obtain any analytic results when analysing a problem on regular lattices, as in percolation theory.

†See also ONISHCHENKO, D. A., Some principles of the construction and analysis of quasistatic models of the probability mechanics of the breakdown of discrete systems. Preprint No. 572, Inst. Problem Mekhaniki Ross. Akad. Nauk, Moscow, 1996.

In this paper, the breakthrough problem will be considered for several forms of a linear chain of lattice nodes and in the case when the structure of the system is a regular binary tree (a Cayley tree). Relations, which determine the breakthrough probability to a specified depth, will be obtained analytically for these cases and certain features of the model will be clarified. It can be assumed that some of the properties of the breakthrough model, obtained for the lattices mentioned above, may be found to be similar to the case of other lattices by virtue of the empirical principle of universality which finds its confirmation in percolation theory [7, 8].

2. SOLUTION OF THE PROBLEM IN THE CASE OF A LINEAR CHAIN OF LATTICE NODES

We will consider the simplest, one-dimensional lattice with a linear configuration of the lattice nodes (Fig. 2). It is required to find the probability that, for a specified value of W, the breakthrough zone reaches a "depth" $k = N$. Such an event is equivalent to the product of the events involving the rupture of lamellae $1, 2, \ldots, N$ under the loads $\Delta p_1, \Delta p_2, \ldots, \Delta p_N$ respectively, where, according to relation (1.1), $\Delta p_k = p_k - p_0 = [W/(kv_0)]p_0$ $(k = 1, \ldots, N)$. Hence, by virtue of the independence of the strengths of the lamellae, the corresponding probability is equal to

$$P_N(W) = \prod_{k=1}^{N} F_s\left(\frac{W}{k}\right) \tag{2.1}$$

(here and henceforth we assume that $p_0 = 1$, $v_0 = 1$, unless otherwise stated).

For small values of N, distribution (2.1) depends very much on the actual form of the function F_s. We will show that, in the asymptotic limit when $N \to \infty$, the behaviour of the distribution $P_N(W)$ possesses a certain universality.

For convenience we will now change to a "specific load": we shall characterize the amount of injected gas by means of the quantity $c = W/N$. Relation (2.1) is then written in the form

$$P_N(c) = \prod_{k=1}^{N} F_s\left(\frac{cN}{k}\right) \tag{2.2}$$

It is clear that $P_N(c) = 0$ when $c \leqslant c_{\min}$ and $P_N(c) = 1$ when $c \geqslant c_{\max}$. We will therefore consider the behaviour of the quantity $P_N(c)$ when $c_{\min} < c < c_{\max}$. Suppose that $c_1 = (c + c_{\max})/2$ and, at the same time, $F(c_1) < 1$. If the depth N is sufficiently large, it can be asserted that $(cN)/k < c_1$ when $k > k_1$, where $k_1 = \gamma N$ $(\gamma = c/c_1 < 1)$. From relation (2.2), we now obtain

$$P_N(c) = \prod_{k=1}^{k_1} F\left(\frac{cN}{k}\right) \cdot \prod_{k=k_1+1}^{N} F\left(\frac{cN}{k}\right) \leqslant 1 \cdot [F(c_1)]^{N-k_1} = [F(c_1)]^{(1-\gamma)N}$$

Consequently, $P_N(c) \to 0$ when $N \to \infty$, that is

$$P_N \to H(s_{\max}) \text{ when } N \to \infty \tag{2.3}$$

where H is a step function. This means that the amount of injected gas, necessary for breakthrough up to a depth N, must be proportional to the breakthrough depth for large values of N

$$W(N) \sim s_{\max} N \tag{2.4}$$

Moreover, this conclusion is independent of the actual form of the distribution F_s.

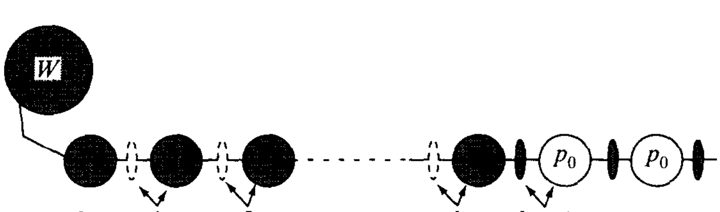

Fig. 2

A somewhat different problem, which is also of interest, can similarly be set up within the framework of this model, that is: to find the distribution of the breakthrough depth as a function of the amount of injected gas and its numerical characteristics (the mean value of the breakthrough depth, for example). Note that the breakthrough depth (which we denote by $D(W)$) is a discrete random quantity.

In the case of a linear chain of lattice nodes, this problem can be solved analytically. We first find the breakthrough probability $P'_k(W)$ *exactly* at a depth k ($k = 0, 1, 2, \ldots$)

$$P'_k(W) = \prod_{i=1}^{k} F_s\left(\frac{W}{i}\right) \cdot \left[1 - F_s\left(\frac{W}{k+1}\right)\right] \tag{2.5}$$

(when $k = 0$, the first factor is assumed to be equal to one). This formula differs from formula (2.1) in that there is an additional factor which is equal to the probability that the lamella with the number $k + 1$ is not destroyed after the rupture of the first k lamellae. The set of quantities $P'_k(W)$ is the distribution series for the random quantity $D(W)$, and the mean value of the breakthrough depth is therefore given by the sum

$$\langle D(W) \rangle = \sum_{k=0}^{\infty} k P'_k(W) \tag{2.6}$$

Similar formulae can be written for the variance and the higher moments.

A rough estimate of the dependence of the quantity $\langle D(W) \rangle$ on W can be obtained using the following arguments. Expression (2.5) vanishes both when $k > W/s_{\min}$ and when $k \leqslant W/s_{\max}$. Hence, the quantity $D(W)$ only takes values in the interval $[W/s_{\max}, W/s_{\min}]$ and this means that the estimate $W/s_{\max} < \langle D(W) \rangle < W/s_{\min}$ holds. Note that it follows from relations (2.3) and (2.4) that $\langle D(W) \rangle \to W/s_{\max}$ as W increases.

3. CERTAIN FEATURES OF THE BREAKTHROUGH MODEL

In order to reveal the distinguishing feature of the breakthrough model compared with models of the percolation type, we will consider a breakthrough problem in the three simple lattices shown in Fig. 3, where the lattice node 0 is the reference lattice node and all links are blocked in the initial state. We call the lattices which are denoted by the letters a, b and c in Fig. 3, lattices of type 1, 2 and 3 respectively. The solution for lattice 1 has been obtained above and is given by formula (2.1).

In order to find the breakthrough probability down to a depth N for lattice 2, we note that, unlike the case of lattice 1, it is now necessary to take account of the multiplicity of configurations of the system corresponding to the breakthrough event. In fact, the breakthrough zone, which reaches a depth N, can include an arbitrary number of "redundant" lattice notes 1′, 3′, ... In the percolation formulation, the fact that the links 0–1′, 2–3′, ... corresponding to these lattice nodes are unblocked has no effect whatsoever on the required probability for the existence of a connecting route from the reference lattice node to a lattice node of the Nth structural level. In the breakthrough model, the unblocking of a link to a redundant lattice node (or to a dead end in percolation terms) leads to a reduction of the load on the lamellae, located on the breakthrough front, which results in a reduction of the breakdown probability for certain links compared with the case when the link leading to the redundant lattice nodes remain blocked. It follows from this that the breakthrough probability in lattice 2 is, generally speaking, less than the breakthrough probability in lattice 1

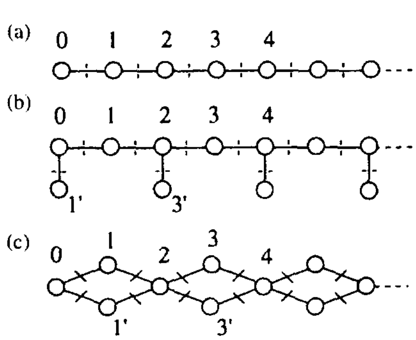

Fig. 3

$$P_N^{(2)}(W) < P_N^{(1)}(W) \tag{3.1}$$

Note that, if a link leading to a redundant lattice node has not been destroyed in a given step, it will not be destroyed later on, and the corresponding lattice node does not therefore enter into the structure of the breakthrough cluster.

We will now calculate the probability $P_N^{(2)}(W)$. We will denote by $H_{i_1 \ldots i_N}$ the configuration of the breakthrough zone, when i_1 lamellae are ruptured in the first structural level, i_2 are ruptured in the second structural level and so on, and, finally, i_N are ruptured in the (Nth) structural level. Here, $i_{2k-1}(k = 1, 2, \ldots)$ can take values of 1 or 2 (since there are two loaded lamellae) and $i_{2k} \equiv 1$. When the breakthrough zone reaches the lattice node with number k, the load q on the active lamellae is equal to $W/(1 + i_1 + \ldots i_{k-1})$. Since the different configurations of the breakthrough cluster, which uniquely define the sets $\{i_1, \ldots, i_N\}$, are mutually exclusive events and the strengths of the lamellae are independent random quantities, the probability of the breakthrough up to a depth N is found as the sum of the probability of the realization of all possible configurations (in accordance with the addition theorem for the probability of a sum of events [12]). When N is even, we obtain

$$P_N^{(2)}(W) = \sum_{i_1=1}^{2} [F_s(q_0)]^{i_1}[1 - F_s(q_0)]^{2-i_1} F_s(q_{i_1})$$

$$\sum_{i_3=1}^{2} [F_s(q_{i_1+i_2})]^{i_2}[1 - F_s(q_{i_1+i_2})]^{2-i_3} F_s(q_{i_1+i_2+i_3})\ldots$$

$$\ldots \sum_{i_{N-1}}^{2} [F_s(q_{i_1+\ldots+i_{N-2}})]^{i_{N-1}}[1 - F_s(q_{i_1+\ldots+i_{N-2}})]^{2-i_{N-1}} F_s(q_{i_1+\ldots+i_{N-1}}) \tag{3.2}$$

where the notation

$$q_j = W/(j + 1) \tag{3.3}$$

has been introduced.

We will now explain that, when $i_1 = 1$, the first two factors in the first sum define the probability of the event involving the breaking of the 0–1 link and the non-breaking of the 0–1′ link, and the third factor is equal to the probability of the breaking of the link 1–2 under a load $W/2$. When $i_1 = 2$, the first two factors in the first sum define the probability of an event involving the simultaneous breaking of the links 0–1 and 0–1′ while the third factor is equal to the probability of the breaking of link 1–2 under a correspondingly smaller load $W/3$. Similar arguments also hold for the remaining sums.

It follows immediately from inequality (3.1) that, in the case of the normalization

$$W = cN$$

the asymptotic property (2.3) is also true in the case of lattice 2. If, however, another normalization of the amount of injected gas

$$W = c(\tfrac{3}{2}N)$$

is considered (for which the amount of injected gas is compared with the total volume of the "pore space" within the limits of N structural levels), then additional investigation is required. It can be proved that property (2.3) also holds in this case (the corresponding calculations are analogous to those presented in Section 2 but more lengthy).

We will now consider lattice 3 (Fig. 3c). It can be obtained from lattice 2 by the addition of the links 1′–2, 3′–4, ... Apart from the further large increase in the number of possible configurations of the breakthrough cluster, this leads to the occurrence of the new of effect of the "retarded" unblocking of the lattice nodes. To explain this consider the unit {0, 1, 2, 1′} in Fig. 3(c). We assume that the link 0–1 is broken in the first step and that the 0–1′ link is not broken, thereby keeping the lattice node 1′ closed. For the subsequent development of the breakthrough process, the breaking of link 1–2 is necessary, which leads to the loading of link 2–1′. In the case of its possible rupture, the lattice node 1′ is unblocked, it is added to the breakthrough cluster and thereby has an effect on the subsequent development of the breakthrough process. It is clear that, in the case of a more complex lattice topology, the "retardation time" can be, generally speaking, arbitrary which greatly complicates (if not completely precluding) the finding of an analytic solution.

In the version of the lattice being considered, a solution is nevertheless successfully found, to be sure, at the cost of introducing a single additional assumption. If it is assumed that, after initially breaking one of the links 1–2 and 1′–2, the second of them is broken (or not broken) and it is only then that the state of the links 2–3 and 2–3′ is analysed (naturally, a similar rule must also be applied in all of the remaining links), then it can be proved that the breakthrough probability in the case of even N is calculated using the formula

$$P_N^{(3)}(W) = \sum_{i_1=2}^{3} \{2f_0(1-f_0)f_1(f_2)^{i_1-2}(1-f_2)^{3-i_1} + (i_1-2)[f_0^2 f_2(2-f_2)]^{i_1-2}\}\dots$$

$$\dots\sum_{i_{N-1}=2}^{3} \{2f_{i_{N-1}}(1-f_{i_{N-1}})f_{i_{N-1}+1}(f_{i_{N-1}+2})^{i_{N-1}-2}(1-f_{i_{N-1}+2})^{3-i_{N-1}} +$$

$$+(i_{N-1}-2)[f_{i_{N-1}}^2 f_{i_{N-1}+2}(2-f_{i_{N-1}+2})]^{i_{N-1}-2}\} \tag{3.4}$$

$$f_{i_k+l} = F_s(q_{i_1+\dots+i_k+l}),\ k=0,\dots,N-1; l=0,1,2;\ i_0=0$$

The quantities q are calculated using relation (3.3); $i_1, i_3, \dots, i_{N-1}$ are the variables of summation in (3.4).

Relation (3.4) is much more complex than (3.2) and it is not possible here to carry out an asymptotic analysis using the same approach as above. The question as to which of the lattices, 2 or 3, is the "stronger" is also open. The answer possibly depends on the form of the distribution F_s.

4. ANALYTIC SOLUTION OF A BREAKTHROUGH PROBLEM ON A CAYLEY TREE

We will now consider the problem of breakthrough on a Cayley tree (the corresponding lattice of nodes is sometimes called a Bethe lattice) with a branching factor of 2 (see Fig. 4, where the scheme for a tree of order $N = 5$ is shown). A special feature of a Cayley tree is the absence of closed paths in its structure. Moreover, if, in conformity with the breakthrough model being considered, the actual breakthrough front is defined as the set of lamellae which are loaded for the first time in a given step (we call such lamellae active lamellae), then it can only move away from the reference lattice node. This makes it much easier to determine the breakthrough depth.

We define the event Z as the formation of at least a single breached channel which reaches a depth N (that is, the formation of a chain of open pores consisting of at least N links) subject to the condition that the amount of injected gas is specified. As an example, the possible configuration accompanying the breakthrough of the system up to a depth of $N = 5$ is shown in Fig. 4, where the solid lines correspond to ruptured lamellae and the dashed lines correspond to unruptured lamellae. Correspondingly, the set of dark points constitutes the breakthrough cluster.

In order to find the probability of the occurrence of the event Z, it is necessary to identify all the different versions of the configuration of the breakthrough zone leading to event Z. We denote by $H_{i_1\dots i_N}$ those versions when i_1 lamellae are ruptured at the first structural level, i_2 lamellae are ruptured at the second structural level, etc. and, finally, i_N lamellae are ruptured at the Nth level. Here, the condition

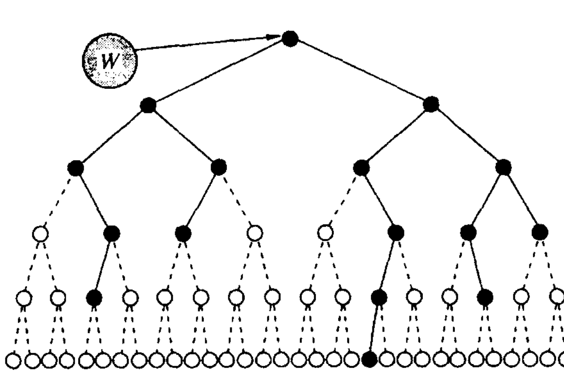

Fig. 4

$i_k \le 2i_{k-1}$ $(k = 1, \ldots, N)$ must be satisfied since a single open pore at the $k-1$ level cannot give more than two open pores of the following structural level.

It can be calculated that there are $C_2^{i_1} C_{2i_1}^{i_2} \ldots C_{2i_{N-1}}^{i_N}$ different configurations of the breakthrough zone in the case of a single set $(i_1, \ldots, i_N)$. Realizations of actual configurations are mutually exclusive events. The inspection of all possible sets $(i_1, \ldots, i_N)$ and the application of the summation theorem leads to the following formula for the probability of breakthrough up to a depth of no less than N

$$P_N(W) = \sum_{i_1=1}^{2} C_2^{i_1} [F_s(q_0)]^{i_1} [1 - F_s(q_0)]^{2-i_1} \sum_{i_2=1}^{2i_1} C_{2i_1}^{i_2} [F_s(q_{i_1})]^{i_2} [1 - F_s(q_{i_1})]^{2i_1 - i_2}$$
$$\sum_{i_3=1}^{2i_2} C_{2i_2}^{i_3} [F_s(q_{i_1+i_2})]^{i_3} [1 - F_s(q_{i_1+i_2})]^{2i_2 - i_3} \ldots$$
$$\ldots \sum_{i_N=1}^{2i_{N-1}} C_{2i_{N-1}}^{i_N} [F_s(q_{i_1+\ldots+i_{N-1}})]^{i_N} [1 - F_s(q_{i_1+\ldots+i_{N-1}})]^{2i_{N-1} - i_N} \tag{4.1}$$

Here $q_j = W/(j+1)$ $(j = 0, 1, \ldots, 2^N)$ is the pressure drop on the lamellae located at the boundary of the breakthrough zone, subject to the condition that exactly $j = i_1 + i_2 + \ldots$ lamellae have been ruptured up to this instant. The products $[\cdot]^{i_k}[\cdot]^{2i_{k-1} - i_k}$ are equal to the probabilities that exactly i_k of the $2i_{k-1}$ loaded lamellae will be ruptured in the following step and that the remaining $2i_{k-1} - i_k$ lamellae will remain intact.

Relation (4.1) is similar to relations (3.2) and (3.4) obtained in the case of a linear chain of pores. However, it has a combinatorial form: in this case, the number of terms (which is equal to the number of different versions of the configurations of the breakthrough zone) increases rapidly as N increases. It is analytically difficult to estimate this dependence effectively. Computer calculations, carried out for small values of N, show that the time taken to calculate the quantity $P_N(W)$ increases on passing from N to $N+1$ by approximately 2^N times. The direct use of formula (4.1) for numerical calculations in the case of large values of N is therefore obviously impossible.

There is considerable interest in the question of the behaviour of the corresponding distribution in the asymptotic limit, that is, as $N \to \infty$. As investigations of a related problem concerning the stochastic strength of a bundle of fibres with a hierarchical structure, which also has the form of a binary Cayley tree (see paper [10] and the references therein), have shown, the limiting distributions can be of different types, including both degenerate step distributions, which correspond to the effect of the onset of a strength threshold (as shown above, such a situation occurs for linear lattices of types 1 and 2), and non-degenerate distributions.

An asymptotic analysis of relation (4.1) is exceedingly complicated, not only within the framework of analytic approaches but, also, as has been noted above, numerical approaches.

Examples of model calculations for $N = 1, 3, 5, 7$ are presented in Fig. 5. It is not the values of W, the amount of injected gas, themselves which have been plotted along the abscissa but the relative values c which are defined by the relation

$$W = cV^{(N)} \tag{4.2}$$

where $V^{(N)} = 2^{N+1}$ is the total volume of the pore space within the limits of N structural levels (for convenience, it is assumed that the volume of the root node is equal to $2v_0$). The traditional Laplace equation [1, 2] for the pressure jump on a lamella

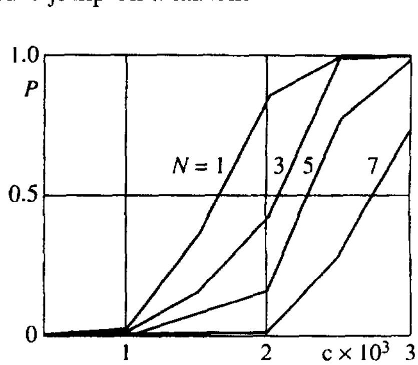

Fig. 5

$$s \sim \gamma / r_t \quad (4.3)$$

is used to fix the strength distribution of the lamellae F_s, where r_t is the radius of the pore throat and γ is the surface tension coefficient (a value of $\gamma = 0.03$ Nm^{-1} was used in the calculations). The radius of the pore throat r_t was taken as the basic, random variable in the calculations. This radius obeys a normal law with a density

$$w_r(x) = \frac{1}{\sqrt{2\pi}\sigma_r} \exp \frac{(x - m_r)^2}{2\sigma_r^2}$$

which is truncated to the interval 10^{-6}–10^{-4} m (the typical range of pore radii [1, 2]). The values $m_r = 5 \times 10^{-5}$ and $\sigma_r = 10^{-5}$ m were taken as the mean value of m_r and the root mean square deviation σ_r.

It follows from relation (4.3) that the strength of the lamellae is confined to the range 300–30000 Pa with the mode (the maximum point of the density function) at the point 600 Pa. Note that the density function is asymmetric and shifted towards smaller values. Atmospheric pressure $p_0 = 10^5$ Pa was taken as the initial pressure.

Graphs of P_N against c, as has already been mentioned above, are actually the strength distribution functions of the fragment of the Cayley tree of order N. The graphs in Fig. 5 show that, as N increases, there is a tendency for the distributions to "drift" to the left, that is, there is a reduction in the strength of the system for a given value of c. We recall that the opposite situation occurred in the cases of the linear structures which were considered above and the distributions were displaced to the right (this is determined by property (2.3)).

5. CONCLUSION

The probability model of the breakthrough of a coarsely cellular foam when an ideal gas is injected into a porous medium differs fundamentally from models of the percolation type. The breakthrough cluster is always bounded and the problem consists of determining the probability of the breakthrough up to a depth no less than a specified depth. The analytical solutions, found in the case of a linear chain of pores and in the case when the structure of the system is a regular binary tree (a Cayley tree) have a very complicated form. It can therefore be assumed that approximate approaches are required for an effective analysis of the breakthrough model in lattices with a more complex structure.

An asymptotic investigation, carried out for a linear chain of nodes, revealed the effect of the onset of a degenerate strength threshold. The limited numerical results, obtained for a Cayley tree, enable us to postulate that, in the case of other lattices, the asymptotic behaviour of the strength distributions of a system of blocking lamellae may turn out to be non-trivial and to depend very much on the method of normalising the amount of injected gas.

The approaches suggested in this paper may find application when investigating models with a different physical content, in particular, when describing a certain process which propagates through a stochastic, tree-like structure from the root to a vertex with gradually decreasing intensity. Problems which arise in the study of breathing difficulties are examples (see [13], where an analysis of a certain mechanical model of forced ventilation of the lungs was carried out using the apparatus of percolation theory).

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REFERENCES

1. ROSSEN, W. R. and GAUGLITZ, P. A., Percolation theory of creation and mobilization of foams in porous media. *AIChE Journal*, 1990, **36**, 1176–1188.
2. ENTOV, V. M. and MUSIN, R. M., Micromechanics of non-linear, two-phase flows in porous media. Network modelling and percolation analysis, *Izv. Ross. Akad. Nauk. MZhG*, 1997, 2, 118–130.
3. KORNEV, K. G., NEIMARK, A. V. and ROZHKOV, A. N., Physical mechanisms of foam flow in porous media. *Advances in the Flow and Rheology of Non-Newtonian Fluids* (Eds D. A. Siginer *et al.*) Elsevier, Amsterdam, 1999, Pt B, 1151–1182.
4. KORNEV, G. G., Capillary foam pinning in porous media. *Zh. Eksper. Teor. Fiz.*, 1995, **107**, 6, 1895–1906.
5. DAUTOV, R., KORNEV, K. and MOURZENKO, V., Foam patterning in porous media. *Phys. Rev. E.*, 1997, **56**, 6929–6944.
6. LARSON, R. G., SCRIVEN, L. E. and DAVIS, H. T., Percolation theory of two phase flow in porous media. *Chem. Eng. sci.*, 1981, **36**, 57–73.

7. EFROS, A. L., The *Physics and Geometry of Disorder*. Nauka, Moscow, 1982.
8. STAUFFER, D. and AHARONY, A., *Introduction to Percolation Theory*. Taylor and Francis, London, 1994.
9. BOLOTIN, V. V., *The Prediction of the Life of Machines and Structures*. Mashinostroyeniye, Moscow, 1984.
10. ONISHCHENKO, D. A., Probability modelling of multiscale fracture. *Izv. Ross. Akad. Nauk. MTT*, 1999, 5, 27–48.
11. YERMAKOV, S. M., *The Monte-Carlo Method and Related Problems*. Nauka, Moscow, 1975.
12. GNEDENKO, V. V., *Course in Probability Theory*. Nauka, Moscow, 1988.
13. SUKI, B., BARABASI, A.-L., HANTOS, Z., PETAK, F. and STANLEY, H. E., Avalanches and power-law behaviour in lung inflation. *Nature*, 1994, **368**, 615–618.

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